Prop 15 Let $(\mathcal{C}_\alpha)_{\alpha \in J}$ be a family of ∞-categories. Then the canonical map

$$h\left(\prod_{\alpha \in J} \mathcal{C}_\alpha\right) \longrightarrow \prod_{\alpha \in J} h(\mathcal{C}_\alpha)$$

is an isomorphism in Cat.

("h commutes with products")

proof: · First, we know from Prop 7 that $\prod \mathcal{C}_\alpha$ is an ∞-category so this makes sense.

· Recall that in Cat, products are formed by taking products of all objects and all morphisms.

· The map of the statement is thus a bijection on objects and surjective on morphisms, so it remains to see that it is injective on morphisms. If

$$([f_\alpha])_{\alpha \in J} = ([g_\alpha])_{\alpha \in J} \text{ in } \prod_\alpha h\mathcal{C}_\alpha,$$

we get a collection of $t_\alpha \in (\mathcal{C}_\alpha)_2$ giving homotopies, but then $(t_\alpha) \in (\prod \mathcal{C}_\alpha)_2$ implies that

$$[(f_\alpha)] = [(g_\alpha)] \text{ in } h\left(\prod_\alpha \mathcal{C}_\alpha\right). \quad \square$$

Cor 16: Let $F: C \to C'$ be an equivalence of ∞-categories. Then $\mathcal{R}F: \mathcal{R}C \to \mathcal{R}C'$ is an equivalence of categories.

proof: Step 1) We show that any natural transformation $\alpha: F \Rightarrow G$ of functor $F, G: C \to D$ induces a natural transformation $\mathcal{R}\alpha: \mathcal{R}F \Rightarrow \mathcal{R}G$.

- We have

$$\mathcal{R}(C \times \Delta^1) \underset{\text{Prop 15}}{\simeq} \mathcal{R}C \times [1]$$

α is by definition a morphism

$$C \times \Delta^1 \longrightarrow D$$

$$\Rightarrow \mathcal{R}\alpha: \mathcal{R}C \times [1] \simeq \mathcal{R}(C \times \Delta^1) \longrightarrow \mathcal{R}D$$

is a natural transformation $\mathcal{R}F \Rightarrow \mathcal{R}G$.

- Step 2) Assume now that α is a natural isomorphism, that is, we have $\beta: G \Rightarrow F$

and

$$t, t' : C \times \Delta^2 \longrightarrow D \quad \text{with}$$

$$\begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\alpha"] \arrow[rr, "id_F"'] \arrow[rr, phantom, shift left=3, "t"] & & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & F \arrow[dr, "\alpha"] & \\ G \arrow[ur, "\beta"] \arrow[rr, "id_G"'] \arrow[rr, phantom, shift left=3, "t'"] & & G \end{tikzcd}$$

Then by using $R(C \times \Delta^2) \underset{\text{Prop 15}}{\simeq} RC \times [2]$

we get that $\begin{cases} R\beta \circ R\alpha \overset{Rt}{=} R\,id_F = id_{RF} \\ R\alpha \circ R\beta \overset{Rt'}{=} R\,id_G = id_{RG} \end{cases}$

so $R\alpha$ is also a natural isomorphism.

<u>Step 3)</u> Assume F is an equivalence of ∞-categories with "inverse" $G : D \to C$. Then we have natural isomorphisms $\begin{cases} F \circ G \overset{\alpha}{\simeq} id_D \\ G \circ F \overset{\beta}{\simeq} id_C \end{cases}$

and by Step 2) : $\begin{cases} RF \circ RG = R(F \circ G) \overset{R\alpha}{\simeq} id_{RD} \\ RG \circ RF = R(G \circ F) \overset{R\beta}{\simeq} id_{RC} \end{cases}$

So hF is an equivalence of 1-categories. $\blacksquare$

⚠ The converse is not at all true!

Ex: K Kan complex.

$$\begin{cases} K \to \Delta^0 \text{ is an eq. of } \infty\text{-categories} \iff K \text{ contractible} \\ hK \simeq \Pi_1 |K| \,;\, hK \to [0] \text{ is an eq} \iff |K| \text{ is connected and simply-connected} \end{cases}$$

def 17 Let C be an ∞-category.

A morphism $f \in C_1$ is an **isomorphism** if $[f] \in \mathrm{Mor}(hC)$ is an isomorphism, i.e if $\exists\, g \in C_1$ with $[f] \circ [g]$ and $[g] \circ [f]$ identities in hC, i.e $\exists\, t, t' \in C_2$

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] \arrow[d, phantom, "t"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, equal] & & \cdot \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \cdot \arrow[dr, "f"] & \\ \cdot \arrow[ur, "g"] \arrow[rr, equal] & & \cdot \end{tikzcd}$$

Ex: $\cdot\ \forall x \in C_0$, $id_x = s_0(x)$ is an iso.

$\cdot$ If C is a 1-category, then isomorphisms in NC are exactly the isos. in C.
$(hNC \xrightarrow{\sim} C)$

def 18: An ∞-groupoid is an ∞-category in which every morphism is an iso.

lemma 19: Kan complexes are ∞-groupoids.

proof: Same argument as part ii) of Prop 4. $\square$
(C 1-cat, C groupoid $\Leftrightarrow$ NC Kan)

The converse is true but much more difficult
we will see (at least parts of) the proof later.

thm 20: ∞-groupoids are Kan complexes.

Rmk: • Try to prove this by hand!
Even proving the lifting property for
$\Lambda^3_0 \hookrightarrow \Delta^3$ is challenging.

(Kan $\overset{\text{def}}{\Leftrightarrow}$ lifting for all $\Lambda^n_k \hookrightarrow \Delta^n$, $0 \leq k \leq n$

∞-groupoid "$\Leftrightarrow$" ———————— $0 < k < n$

+ ———————— $\Lambda^2_0 \hookrightarrow \Delta^2$, $\Lambda^2_2 \hookrightarrow \Delta^2$)

Rmk: Thm 20 implies that the (not yet defined) ∞-category of ∞-groupoids is equivalent to the ∞-category of Kan complexes, which by simplicial homotopy theory is equivalent to the ∞-cat. of topological spaces. $\Rightarrow$ Grothendieck's homotopy hypothesis holds in the quasicategory model of $(\infty,1)$-categories.

- Another important theorem, which will be our main goal in the next section of the course, is

thm 21: Let $K \in sSet$, $C \in Cat_\infty^1$.

Then the simplicial set $Fun(K,C)$, with $Fun(K,C)_n = sSet(K \times \Delta^n, C)$, is an ∞-category. $\square$

- C, D ∞-categories. Then

  $Fun(C,D)_0 = \{\text{functors } C \to D\}$

  $Fun(C,D)_1 = \{\text{natural transformations } C \times \Delta^1 \to D\}$

- In fact, the definition of natural isomorphisms and categorical equivalences can be reformulated as:

- $\begin{pmatrix} \alpha : F \Rightarrow G \\ \text{natural iso} \end{pmatrix} \iff \begin{matrix} \alpha : F \to G \text{ isomorphism} \\ \text{in } Fun(C,D) \end{matrix}$

- $\begin{pmatrix} C \xrightarrow{F} D \\ \text{categorical} \\ \text{equivalence} \end{pmatrix} \iff \begin{pmatrix} \exists\, G : D \to C, \\ F \circ G \text{ iso. to } id_D \text{ in } Fun(D,D) \\ G \circ F \text{ ——— } id_C \text{ in } Fun(C,C) \end{pmatrix}$

. Another application of the homotopy category is to the notion of subcategory:

def 22: A subcategory C' of an ∞-category C is a simplicial subset which satisfies moreover: $\forall n \geq 2$, $\forall \gamma \in C_n$,

$$\gamma \in C'_n \iff \gamma_{|I^n} \text{ has edges in } C'_1.$$

Hence C' is determined by C'_0 and C'_1.

C' is a full subcategory if

$\forall n \geq 1$, $\forall \gamma \in C_n$,

$\gamma \in C'_n \iff$ the vertices of γ lie in C'_0.

C' is then determined entirely by C'_0.

lemma 23: A subcategory C' of an ∞-category C is an ∞-category.

proof: This follows immediately from $I^n \subseteq \Lambda^n_k$.

$$\begin{tikzcd}
I^n \arrow[r, hook] \arrow[dr, bend right] & \Lambda^n_k \arrow[r] \arrow[d, hook] & C' \arrow[r, hook] & C \\
& \Delta^n \arrow[ur, dashed] \arrow[urr, dashed, "\gamma"'] & &
\end{tikzcd}$$

$\square$

Prop 24: Let C be an ∞-category and $\eta : C \longrightarrow N h C$ be the unit map.

Then a) If D is a subcategory of the 1-category hC, then the pull back $\eta^{-1}(ND) := ND \times_{NhC} C$ is a subcategory of C.

$$\begin{tikzcd} \eta^{-1}(ND) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\eta"] \\ ND \arrow[r, hook] & NhC \end{tikzcd}$$

b) This construction gives a bijection

$$\{\text{Subcategories of } hC\} \simeq \{\text{Subcategories of } C\}$$

which restricts to a bijection of full subcategories.

proof: We only do the case of subcategories, the full case is easier.

a)

Claim | If C is a 1-category and C' is a (full) subcategory of C, then NC' is a (full) subcategory of NC.

pf: obvious from the unique lifting property of nerves along $I^n \hookrightarrow \Delta^n$.

Claim: | the pullback of a subcategory is a subcategory.

pf: Let $C' \subseteq C$ be a subcategory of an ∞-cat. and $E \xrightarrow{f} C$ be any morphism. Then $E' := E \times_C C'$ is a simplicial subset of E $\left(\begin{array}{l}\text{pullback of mono is}\\ \text{mono in any category}\end{array}\right)$

By definition, we have for $n \in \mathbb{N}$: $E'_n = E_n \times_{C_n} C'_n$, so for $z \in E_n$, we have:

$$z \in E'_n \overset{\text{pullback}}{\iff} f(z) \in C'_n$$

$$\overset{\text{subcategory}}{\iff} f(z)_{|I^n} \text{ has edges in } C'_1$$

$$\iff f(z_{|I^n}) \text{ ——— } C'_1$$

$\overset{\text{pullback}}{\Longleftrightarrow} \quad \exists!\, I^n \longrightarrow E_1'$

so that E' is a subcategory.

Together the two claims imply a).

b) <u>Injectivity:</u>

The morphism $C \longrightarrow N\mathfrak{R}C$ induces

- a bijection $C_0 \xrightarrow{\sim} (N\mathfrak{R}C)_0$
- a surjection $C_1 \twoheadrightarrow (N\mathfrak{R}C)_1$.

So if D is a subcategory of $\mathfrak{R}C$, then

$\eta^{-1}\mathfrak{R}D$ determines $\begin{cases} Ob(D) = \eta\left((\eta^{-1} N D)_0\right) \\ Mor(D) = \eta\left((\eta^{-1} N D)_1\right) \end{cases}$

hence determines D.

This proves the injectivity.

<u>Surjectivity</u>: Let $C' \subseteq C$ be a subcategory.

The natural candidate for D is hC', so let's try that! We first check that $hC' \to hC$ is a subcategory. We have $Ob(hC') = C'_0 \hookrightarrow C_0 = Ob(hC)$.

<u>Q</u>: $Mor(hC') = C'_1/_{homotopy} \overset{?}{\hookrightarrow} C_1/_{homotopy} = Mor(hC)$

If $f, g \in C'_1$ are homotopic in C, it means there exists
$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & {} \arrow[u, phantom, "t"] & y \end{tikzcd} \in C_2$$

but $t \in C'_2 \subset C_2$ because f, id_y lie in C'_1. and C' is a subcategory.

$\Rightarrow$ hC' is a subcategory of hC.

- It remains to show that
$$\begin{tikzcd} C' \arrow[r] \arrow[d, "\eta"'] & C \arrow[d, "\eta"] \\ NhC' \arrow[r] & NhC \end{tikzcd}$$

is a pullback square. On n-simplices, this says that

$$z \in C'_n \overset{?}{\iff} \text{“the edges in } z_{|I^n} \text{ are homotopic in } C \text{ to morphisms in } C'\text{.”}$$

But as we saw, this last condition is $\iff$ "the edges of $z_{|I^n}$ are morphisms of C'_1"

which by definition of a subcategory is precisely $\iff z \in C'_n$. $\square$

Ex: • If $C \in Cat$, then

$$\left\{ \text{(full) subcategories of } C \right\} \xrightarrow{\sim} \left\{ \text{(full) subcategories of } NC \right\}$$

$$D \longmapsto ND$$

Rmk: The notion of subcategory in Cat is not invariant under equivalences.

Replete subcategories in C = C' subcategory such that $X \in C$, $X \simeq Y \in C'$ $\Rightarrow X \in C'$.

- To finish this introductory section, we need to see at least one example which does not come from nerves or Kan complexes.

The idea is that, in the same way that the nerve N gives

$$N : \{1-\text{categories}\} \subset \{\infty-\text{categories}\}$$

there should exist a "fully faithful" functor:

$$N_2 : \{(2,1)-\text{categories}\} \subset \{\infty-\text{categories}\}$$

This does exist! However:

Pb: The definition of general (weak) (2,1)-categories is complicated!

Even strict (2,1)-categories require

ideas that I don't want to discuss (yet).

Sol: We know <u>one</u> example of a strict $(2,1)$-category Cat^2:

$$\begin{cases} \mathrm{Ob}\ \mathrm{Cat}^2 : \text{ small categories} \\ \mathrm{Mor}\ \mathrm{Cat}^2 : \text{ functors} \\ 2\text{-}\mathrm{Mor}\ \mathrm{Cat}^2 : \text{ natural isomorphisms.} \end{cases}$$

So the plan is to construct an ∞-category $N_2(\mathrm{Cat}^2)$ from this; the recipe then generalizes to any weak $(2,1)$-category (N_2 is called the Duskin nerve).

Example: We define a simplicial set $N_2(\mathrm{Cat}^2)$ follows: an n-dimensional element in $(N_2(\mathrm{Cat}^2))_n$ is the datum of:

- $\forall\ 0 \leq i \leq n$, a small category C_i
- $\forall\ 0 \leq i \leq j \leq n$, a functor $F_{i,j}: C_i \longrightarrow C_j$

- $\forall\, 0 \le i \le j \le k \le n$, a natural iso. $\alpha_{i,j,k} : F_{i,k} \Rightarrow F_{j,k} F_{i,j}$

such that:
- $F_{i,i} = id_{C_i}$
- $\alpha_{i,i,j}$ and $\alpha_{i,j,j}$ are identities.
- $\forall\, 0 \le i \le j \le k \le l \le n$,

the diagram

$$\begin{tikzcd} F_{i,l} \arrow[r, Rightarrow, "\alpha_{i,j,l}"] \arrow[d, Rightarrow, "\alpha_{i,k,l}"'] \arrow[dr, phantom, "(*)"] & F_{j,l} F_{i,j} \arrow[d, Rightarrow, "\alpha_{j,k,l}"] \\ F_{k,l} F_{i,k} \arrow[r, Rightarrow, "\alpha_{i,j,k}"] & F_{k,l} F_{j,k} F_{i,j} \end{tikzcd}$$

commutes.

- For $\delta : [m] \longrightarrow [n]$, we define

$$\delta^*(C_i, F_{i,j}, \alpha_{i,j,k}) = (C_{\delta(i)}, F_{\delta(i),\delta(j)}, \alpha_{\delta(i),\delta(j),\delta(k)})$$

Rmk: $N\,\mathrm{Cat}$ is isomorphic to the simplicial subcomplex of $N_2(\mathrm{Cat}^2)$ on the elements where

$$F_{i,k} = F_{j,k} F_{i,j} \quad \text{and} \quad \alpha_{i,j,k} = id.$$

So we have added some new higher morphisms to $N\,\mathrm{Cat}$, corresponding to non-id. natural isomorphisms.

prop 25: $N_2(Cat^2)$ is an ∞ - category.

proof: • The proof is similar to the proof that the nerve of a category is an ∞-category; we just need to go "one dimension higher".

• By definition of $N_2(Cat^2)$, an n-simplex is determined by its restriction to $Sk_3(\Delta^n)$ ("Cat^2 is 3-coskeletal"). Since we have

$$Sk_3(\Lambda^n_k) = Sk_3(\Delta^n) \quad \text{for all } n \geq 4,$$

we only need to check the inner horn liftings for $n \leq 3$.

n=2 A map $\Lambda^2_1 \longrightarrow N_2(Cat^2)$ is just a datum

$$\begin{tikzcd} & C_1 \arrow[dr, "F_{12}"] & \\ C_0 \arrow[ur, "F_{01}"] & & C_2 \end{tikzcd}$$

which we fill up with

$$\begin{tikzcd} & C_1 \arrow[dr, "F_{12}"] \arrow[d, phantom, "\parallel"] & \\ C_0 \arrow[ur, "F_{01}"] \arrow[rr, "F_{12}F_{01}"'] & {} & C_2 \end{tikzcd}$$

Note: this is not unique in general, we can take any natural iso $G \simeq F_{12}F_{01}$.

<u>**$n=3$**</u> A map $\Lambda^3_1 \longrightarrow N_2(\mathcal{Cat}^2)$ is

a diagram:

$$\begin{tikzcd}
C_0 \arrow[dd, "F_{01}"'] \arrow[dr, "F_{02}"] \arrow[drr, bend left, "F_{03}"] & & \\
C_1 \arrow[u, phantom, "\overset{\alpha_{012}}{\underset{\sim}{\Rightarrow}}"] & C_2 \arrow[r, "F_{23}"] \arrow[u, phantom, "\wr\Downarrow\alpha_{023}"] \arrow[d, phantom, "\wr\Uparrow\alpha_{123}"] & C_3 \\
C_1 \arrow[ur, "F_{12}"] \arrow[urr, bend right, "F_{13}"'] & &
\end{tikzcd}$$

By the condition ⊛ in the definition,

we must fill this in with

$$\begin{tikzcd}
C_0 \arrow[d, "F_{01}"'] \arrow[dr, bend left, "F_{03}"] \arrow[dr, phantom, "\Downarrow\alpha_{013}"] & \\
C_1 \arrow[r, bend right, "F_{13}"'] & C_3
\end{tikzcd}$$

with $\alpha_{013} = \alpha_{123}^{-1} \circ \alpha_{012} \circ \alpha_{023}$ (2,1)-cat.

and it works. $\square$

III Lifting calculus

- The definition of ∞-categories is very combinatorial and the proofs so far have been by explicit manipulations of simplices.
- To go further and prove results like Thm 20-21, we need a more systematic way to keep the combinatorics manageable.

1) Fibrations and anodyne maps

Let us formalize a concept we have already seen a lot:

def 1: . Let C be a category. A lifting problem in C is a commutative square in C:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "p"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

- A $\left\{\begin{array}{l}\text{solution}\\ \text{lift}\end{array}\right.$ of the lifting problem is a diagonal map:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\equiv", near start] & X \arrow[d, "p"] \\ B \arrow[r, "v"] \arrow[ur, dashed, "R"] \arrow[ur, phantom, "\equiv", near end] & Y \end{tikzcd}$$

- If S, T are collections of morphisms in $\mathcal{C}$, we say that S has the left lifting property with respect to T

or equivalently that

T has the right lifting property with respect to S.

and write $S \boxslash T$ if every lifting problem as above with $g \in S$ and $p \in T$ has a solution.

In particular, if $S = \{g\}$ (resp. $T = \{p\}$) is a singleton, we say that g has the left lifting property w.r.t to T (resp. p has the right lifting property w.r.t. to S)

and write $g \boxslash T$ (resp. $S \boxslash p$).

- We define the right complement of S, resp. the left complement of T, by

$$S^{\boxslash} := \{ p \in \mathrm{Mor}(C) \mid i \boxslash p \text{ for all } i \in S \}$$

$$\left(\text{resp. } {}^{\boxslash}T := \{ i \in \mathrm{Mor}(C) \mid i \boxslash p \text{ for all } p \in T \}\right.$$

[alternative notations: $RLP(S)$ / $LLP(T)$]

$S \perp T$ ($\Leftrightarrow S \boxslash T$)

Ex: in sSet, we have already several examples of this pattern:

- $\{\text{Kan fibrations}\} := \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k \leq n \right)^{\boxslash}$
- $\{\text{trivial Kan fibrations}\} := \left(\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 1 \right)^{\boxslash}$
- $C \in$ sSet is an ∞-category iff the map $C \to \Delta^0$ lies in $\left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n \right)^{\boxslash}$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & C \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

- On the other hand, the Grothendieck-Segal condition for $X \simeq N(C)$ with $C \in$ Cat cannot be expressed this way because it requires imposing the <u>uniqueness</u> of solutions.

Ex: Some other examples you may be familiar with:

- $\underline{\mathcal{C} = \mathrm{Set}}$: $(\text{injections}) \boxslash (\text{surjections})$

  and in fact:

$$\begin{cases} (\text{inj.}) = {}^{\boxslash}(\text{surj.}) = {}^{\boxslash}(* \amalg * \to *) \\ (\text{surj.}) = (\text{inj.})^{\boxslash} = (\emptyset \to *)^{\boxslash} \end{cases}$$

- $\underline{\mathcal{C} = \mathrm{Ab} \text{ (or any abelian category)}}$:

  - $I \in \mathrm{Ab}$ is called *injective* if the unique map $I \to 0$ lies in $(\text{mono.})^{\boxslash}$.

  - $P \in \mathrm{Ab}$ is called *projective* if the unique map $0 \to P$ lies in ${}^{\boxslash}(\text{epi})$.

- $\underline{\mathcal{C} = \mathrm{Top}}$:

  A morphism $p : X \to Y$ is called a *Serre fibration* if $p \in (D^n \hookrightarrow D^n \times I)^{\boxslash}$

a Hurewicz Fibration if $p \in (X \hookrightarrow X \times I \mid X \in Top)^{\boxslash}$.

Lemma 2: With the notations of def 1:

a) $R \subseteq S \Rightarrow \begin{cases} S^{\boxslash} \subseteq R^{\boxslash}, \text{ and} \\ {}^{\boxslash}S \subseteq {}^{\boxslash}R. \end{cases}$

b) $S \subseteq {}^{\boxslash}(S^{\boxslash})$ and

$$S \subseteq ({}^{\boxslash}S)^{\boxslash}.$$

c) $S^{\boxslash} = ({}^{\boxslash}(S^{\boxslash}))^{\boxslash}$ and

$${}^{\boxslash}S = {}^{\boxslash}(({}^{\boxslash}S)^{\boxslash}).$$

proof: · a) is clear: we have more lifting problems to solve with S than with R.

b) By duality, we only need to prove the first.

- $S \subseteq {}^{\boxslash}(S^{\boxslash})$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "S \ni"'] & X \arrow[d, "\in S^{\boxslash}"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

c) Again by duality we prove only the first:

$$\begin{cases} S^{\boxslash} \overset{b)}{\subseteq} \left({}^{\boxslash}(S^{\boxslash})\right)^{\boxslash} \\ \left({}^{\boxslash}(S^{\boxslash})\right)^{\boxslash} \overset{a)+b)}{\subseteq} S^{\boxslash} \end{cases}$$

$\square$

---

def 3: We define collections of morphisms in sSet:

- $(\text{inner fibrations}) := \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n\right)^{\boxslash}$

- $(\text{left fibrations}) := \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k < n\right)^{\boxslash}$

- $(\text{right fibrations}) := \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k \leq n\right)^{\boxslash}$

$$\Big[\cdot \ (\text{Kan fibrations}) = \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k \leq n\right)^{\boxslash} \Big]$$

$$\cdot \begin{pmatrix} \text{inner anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{inner fibrations})$$

$$\cdot \begin{pmatrix} \text{left anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{left fibrations})$$

$$\cdot \begin{pmatrix} \text{right anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{right fibrations})$$

$$\cdot \begin{pmatrix} \text{anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{Kan fibrations})$$

Rmk: . From the definition and Lemma 2:

$$\begin{tikzcd}
& \text{Kan fibration} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{left fibration} \arrow[dr, Rightarrow] & & \text{right fibration} \arrow[dl, Rightarrow] \\
& \text{inner fibration} &
\end{tikzcd}$$

We concentrate → inner fibration

on these in this chapter.

$$\begin{tikzcd}[column sep=tiny]
 & \text{inner anodyne} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{left anodyne} \arrow[dr, Rightarrow] & & \text{right anodyne} \arrow[dl, Rightarrow] \\
 & \text{anodyne} &
\end{tikzcd}$$

- Moreover, it is easy to see that

  { Kan fibration $\Leftrightarrow$ { left and right fibration.

(⚠ { inner anodyne $\not\Leftrightarrow$ { left and right anodyne.)

- $X \in sSet$,

  X ∞-category $\Leftrightarrow$ $X \to \Delta^0$ inner fibration.

- We will see later that any functor $C \xrightarrow{F} D$ of ∞-categories is an inner fibration "up to equivalence":

$$\begin{tikzcd}
C \arrow[rr, "F"] \arrow[dr, "\sim", "\text{eq.}"'] & \arrow[d, phantom, "\equiv"] & D \\
& C' \arrow[ur, "\text{inner fibration.}"'] &
\end{tikzcd}$$

- "anodyne" is due to Gabriel and Zisman (1967) and means $\sim$ "without pain" in Greek. Central notion in simplicial homotopy theory, the left/right/inner versions are natural extensions in the context of quasicategories.

Lemma 4: Let $X \in sSet$ and $C \in Cat$. Then $X \longrightarrow NC$ is an inner fibration iff X is an ∞-category.

In particular, if $F: C \to D$ is a functor of 1-categories, then $NF: NC \to ND$ is an inner fibration.

proof: Exercise. Hint: use the $\tau \dashv N$ adjunction and consider the maps $\tau(\Lambda^n_k) \longrightarrow \tau(\Delta^n)$ for $0 < k < n$

$\square$

Collections of morphisms of the form ${}^{\boxslash}S$ (or $S^{\boxslash}$) have some remarkable "closure" properties.

def 5: Let $S \subseteq \mathrm{Mor}(\mathcal{C})$. We say that:

* S is closed under pushouts if for every pushout diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & A' \arrow[d] \\ B \arrow[r] & B' \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$
, $g \in S \Rightarrow g' \in S$.